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New S-dualities in $N = 2$ supersymmetric $\mathrm{SU}(2) \times \mathrm{SU}(2)$ gauge theory

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Abstract: New S-dualities in a scale invariant $N = 2$ gauge theory with $\mathrm{SU}(2)\times\mathrm{SU}(2)$ gauge group are derived from embeddings of the theory in two different larger asymptotically free theories. The true coupling space of the scale invariant theory is a 20-fold identification of the coupling space found in the M- and string-theory derivations of the low energy effective action, implying a larger S-duality group. Also, this coupling space is different from the naively expected direct product of two $\mathrm{SL}(2, \mathbf{Z})$ fundamental domains, as it contains a different topology of fixed points.

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1. Introduction and summary

One of the most striking elements in recent developments in our understanding of gauge theories and string theories is the ubiquitous appearance of S-dualities in theories with 8 or more supercharges. S-duality denotes the exact equivalence of a theory at weak coupling to another theory at strong coupling. It can be described in general as a set of identifications on the space of couplings of a theory (or theories). Well-known examples in four-dimensional field theory are the $N=4$ supersymmetric Yang-Mills theories [1] and finite $N=2$ theories [2]–[8]. In $N=4$ theories, for example, S-duality identifies theories with gauge couplings τ and $-1/\tau$.

The S-dualities of classes of scale invariant $N=2$ gauge theories with simple gauge groups [9] and product gauge groups [10] were derived by embedding those theories in higher rank asymptotically free gauge theories. The coupling space of the scale invariant theory was realized as a submanifold of the Coulomb branch of the asymptotically free theory. These embedding arguments by themselves do not necessarily capture all possible S-dualities — there may be further identifications of the coupling space — since they only show that a submanifold of the Coulomb branch of the appropriate asymptotically free theory is some multiple cover of the true coupling space. One place where we know such further identifications must exist are in theories with SU(2) gauge group factors: for in the limit that the other factors decouple, the remaining SU(2) factor must have the full $\mathrm{SL}(2,\mathbb{Z})$ duality of [2], rather than the subgroup $\Gamma^0(2) \subset \mathrm{SL}(2,\mathbb{Z})$ which emerges from the embedding argument. The purpose of this letter is to explore these further S-dualities in a scale invariant $N=2$ gauge theory with $\mathrm{SU}(2)\times\mathrm{SU}(2)$ gauge group.

The specific theory we focus on has massless hypermultiplets in the representations $(\mathbf{2},\mathbf{2})\oplus(\mathbf{2},\mathbf{1})\oplus(\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{2})\oplus(\mathbf{1},\mathbf{2})$ of $\mathrm{SU}(2)\times\mathrm{SU}(2)$. It has two exactly marginal complex gauge couplings, τ_1 and τ_2, which are conveniently parameterized by $f_k = e^{i\pi\tau_k}$ (so that weak coupling is at $f_k = 0$). The new S-dualities we find act as a 20-fold identification on $\mathbb{C}^2 \simeq \{f_1, f_2\}$, and are described explicitly in eqns. (4.3)–(4.6) below. The resulting coupling space has a single $\mathbb{Z}_3$ orbifold fixed point, complex lines of $\mathbb{Z}_2$ orbifold fixed points intersecting in an S_3 orbifold point, and no further strong coupling singularities. The weak coupling singularities have the expected structure: in the limit that one coupling vanishes, the S-duality group acts as $\mathrm{SL}(2,\mathbb{Z})$ on the other coupling; nevertheless, the total coupling space is not simply the cartesian product of two $\mathrm{SL}(2,\mathbb{Z})$ fundamental domains.

This paper is organized as follows. In the next section we review the proof of the S-duality of the SU(2) gauge theory [9], clarifying in what sense the $\mathrm{SL}(2,\mathbb{Z})$ group of identifications on the coupling space can be recovered. In section 3 we study the low energy effective action on the Coulomb branch of our scale invariant $\mathrm{SU}(2)\times\mathrm{SU}(2)$ theory. We derive two different forms of the curve encoding this effective action by embedding the theory in either an $\mathrm{SU}(n)\times\mathrm{SU}(n)$ or an $\mathrm{SU}(2n)\times\mathrm{Sp}(2n)$ theory. Demanding that the resulting curves describe equivalent low energy physics implies a non-trivial mapping between the coupling parameters that appear in each description. In section 4 we use this mapping and the results of [10] to prove that there are the "extra" S-duality identifications described above.

2. Deriving the $\mathrm{SL}(2,\mathbb{Z})$ duality of the $\mathrm{SU}(2)$ theory

The $N=2$ theory with $\mathrm{SU}(2)\simeq\mathrm{Sp}(2)$ gauge group and four massless fundamental hypermultiplets is a scale invariant theory with an exactly marginal coupling, the complex gauge coupling τ, taking values in the classical coupling space $\mathcal{M}_{cl} = \{\tau | \mathrm{Im}\tau > 0\}$. In [2] evidence was presented, in the form of the invariance of the low energy effective action, that the true coupling space of this theory should be the classical space further identified under the transformations $T : \tau \to \tau + 1$ and $S : \tau \to -1/\tau$. This gives the coupling space as $\mathcal{M} = \mathcal{M}_{cl}/\,\mathrm{SL}(2,\mathbb{Z})$, and $\mathrm{SL}(2,\mathbb{Z})$ is said to be the S-duality group of the theory.[1]

[1] We only discuss the S-duality action on marginal couplings and not on masses or other operators, and so will ignore the distinction between $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{PSL}(2,\mathbb{Z})$ in what follows.

On the other hand, the duality identifications manifest in the low energy effective action of this SU(2) gauge theory derived from either the M-theory construction of [5] or the geometrical engineering of [7] do not comprise the full $\mathrm{SL}(2,\mathbb{Z})$ S-duality group conjectured in [2]. It was shown in [9] that the true coupling space of the scale invariant SU(2) gauge theory can be derived from its different covering spaces represented by submanifolds of Coulomb branches of two different embeddings of this theory in higher rank asymptotically free theories. In this section we review this argument and clarify the relation between the geometry of the covering of the coupling space and the S-duality group.

Consider first the scale invariant SU(2) theory with four massless hypermultiplets in the fundamental representation. The Coulomb branch of the theory is described by the curve [2, 3]

$$y^2 = (v^2 - u)^2 - 4fv^4 \,, \tag{2.1}$$

parameterized by the gauge coupling f and the gauge invariant adjoint vev u, a local coordinate on the Coulomb branch. f is a function of the coupling such that $f \sim e^{i\pi\tau}$ at weak coupling.[2] Embedding this theory into the asymptotically free SU(3) model with 4 quarks and scaling on the Coulomb branch of the latter (while tuning appropriately the masses of the quarks) to the scale invariant SU(2) theory one identifies [9] the coupling space $\mathcal{M}_{\mathrm{SU}} = \{f\}$ with $\mathbb{P}^1$ with two punctures and an orbifold point: a weak coupling singularity $f = 0$, an "ultra-strong" coupling point at $f = 1/4$, and a $\mathbb{Z}_2$ orbifold singularity at $f = \infty$.

On the other hand, this scale invariant $N = 2$ SU(2) gauge theory can be thought of as an Sp(2) theory with 4 massless fundamental flavors, whose curve reads [4]

$$y^2 = x(x - v)^2 - 4gx^3 \,. \tag{2.2}$$

The coupling space $\mathcal{M}_{\mathrm{Sp}} = \{g\}$ of this theory was derived in [9] from its embedding in asymptotically free Sp(4) theory with 4 massless hypermultiplets by tuning on the Coulomb branch of the latter to the scale invariant Sp(2) theory. One then finds that $\mathcal{M}_{\mathrm{Sp}}$ is again the complex manifold $\mathbb{P}^1$ with two punctures and an orbifold point: a weak coupling singularity at $g = 0$, an "ultra-strong" singularity at $g = 1/4$, and a $\mathbb{Z}_2$ orbifold singularity at $g = \infty$.

Both the SU(2) and the Sp(2) descriptions of the scale invariant theory must describe the same physics. In particular, their low energy effective actions described by the complex structure of the curves (2.1) and (2.2) must be the same. We therefore look for an $\mathrm{SL}(2,\mathbb{C})$ transformation on x which maps the zeros of the right sides to one another. Of the six distinct such mappings only two map weak coupling to weak

[2] In the $N = 2$ theories discussed here it is convenient to define the coupling as $\tau = \frac{\vartheta}{\pi} + i\frac{8\pi}{g^2}$, differing by a factor of two from the usual definition.

coupling, and imply the identification

$$f = \frac{4\sqrt{g}(1+2\sqrt{g})}{(1+6\sqrt{g})^2}\,. \tag{2.3}$$

Choosing different signs of the square root gives two maps between $\mathcal{M}_{\rm Sp}$ and $\mathcal{M}_{\rm SU}$, which induce the nontrivial identification $\mathcal{U}$ on $\mathcal{M}_{\rm SU}$

$$\mathcal{U}(f) = \frac{\gamma+2}{(\gamma+3)^2}\,, \tag{2.4}$$

where γ is a root of

$$0 = f\gamma^2 + \gamma + 1\,. \tag{2.5}$$

This gives two maps from $\mathcal{M}_{\rm SU}$ to itself, one for each γ satisfying (2.5). Thus these identifications imply at least a three-fold identification on $\mathcal{M}_{\rm SU}$ (the original point and its two images). In fact, a little algebra shows that the orbit of a generic point under $\mathcal{U}$ is just this set of three points, so $\mathcal{M}_{\rm SU}$ is a triple cover of the true coupling space of the scale invariant SU(2) theory. In particular, the identifications (2.4) map the "strong coupling" point $f = 1/4$ to the $f = 0$ weak coupling singularity, and map the $\mathbb{Z}_2$ point $f = \infty$ to the point $f = 2/9$. In addition, there is a new fixed point under these identifications, namely $f = 1/3$, which it is easy to check is a $\mathbb{Z}_3$ orbifold point. The net result is that with these further identifications, the coupling space becomes topologically a sphere with three special points: the weak coupling puncture (the image of $f = 0$ or $1/4$), a $\mathbb{Z}_2$ orbifold point (the image of $f = 2/9$ or ∞), and a $\mathbb{Z}_3$ orbifold point (the image of $f = 1/3$). Since the map (2.4) is analytic, the true coupling space inherits a complex structure from that of the punctured f-sphere. The order of the orbifold points reflects the nature of the singularity in the complex structure at the punctures.

This argument shows that there are indeed more identifications on the coupling space than were apparent in either the SU(2) form of the curve (2.1) or the Sp(2) form of the curve (2.2). But it might not be clear from this argument how to actually see the SL$(2,\mathbb{Z})$ structure of the duality group. For this we need an intrinsic definition of what we mean by duality group. Since having an S-duality group Γ means that the coupling space is given by $\mathcal{M} = \mathcal{M}_{cl}/\Gamma$, and the classical coupling space $\mathcal{M}_{cl}$ is simply connected, we can define [5]

$$\Gamma = \pi_1(\mathcal{M})\,. \tag{2.6}$$

When $\mathcal{M}$ has orbifold singularities, $\pi_1(\mathcal{M})$ should be understood in the orbifold sense [5], meaning that the generator U of $\pi_1(\mathcal{M})$ corresponding to looping about a $\mathbb{Z}_n$ orbifold point satisfies $U^n = 1$.

The true SU(2) coupling space deduced above has the complex structure of a sphere with one puncture, a $\mathbb{Z}_2$ orbifold point, and a $\mathbb{Z}_3$ orbifold point. Thus the S-duality group $\pi_1(\mathcal{M})$ has two generators which we can take to be U, generating loops around the $\mathbb{Z}_2$ point, and V, generating loops around the $\mathbb{Z}_3$ point, and satisfying $U^2 = V^3 = 1$. There are no other constraints since we know that going around the weak coupling puncture is a θ-angle rotation, which does not correspond to any orbifold identification. But $\mathrm{SL}(2,\mathbb{Z})$, considered as an abstract infinite discrete group, can be presented as the group with two generators S and T satisfying only the relations $S^2 = (ST)^3 = 1$. So, identifying $S = U$ and $ST = V$, we see that the S-duality group is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$.

3. Curves for the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ theory

In preparation for our discussion of S-duality in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ scale invariant theory, we must first make a somewhat lengthy technical detour to derive useful forms for the curves whose complex structure encodes the low energy physics of the Coulomb branch of the theory. The different curves we need are those arising from viewing the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ theory as part of an $\mathrm{SU}(n) \times \mathrm{SU}(n)$ series or as part of an $\mathrm{SU}(2n) \times \mathrm{Sp}(2n)$ series. The goal of this section is to derive an explicit map between the couplings of the two versions of the theory — the analog of eq. (2.3) above. This map is summarized at the end of this section for those who prefer to skip the technicalities.

We start by briefly reviewing the derivation [5] of the $\mathrm{SU} \times \mathrm{SU}$ curves from an M5 brane configuration in M-theory. In section 3.2 we then derive curves for the $\mathrm{SU} \times \mathrm{Sp}$ series with fundamental matter using an M5 brane configuration on $\mathbb{R}^7 \times Q$ where Q is the Atiyah-Hitchin manifold, corresponding to a negatively charged O6 orientifold in a type IIA string picture. In section 3.3 we specialize to vanishing bare masses for the matter hypermultiplets in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{SU}(2) \times \mathrm{Sp}(2)$ curves, develop hyperelliptic forms for both curves, and then derive the mapping of parameters matching the two. In section 3.4 we summarize the results of this section relevant for our discussion of S-duality.

3.1 $\mathrm{SU} \times \mathrm{SU}$ curves

Consider the scale invariant $\mathrm{SU}(n) \times \mathrm{SU}(n)$ theory with one hypermultiplet in the bifundamental, n in the first $\mathrm{SU}(n)$ fundamental, and n in the second $\mathrm{SU}(n)$ fundamental. This can be realized as a IIA brane configuration by placing three NS5 branes along the $x^{0\cdots5}$ directions separated in x^6 but located at equal values in $x^{7\cdots9}$, and n D4 branes along the $x^{0\cdots3}$ and x^6 directions suspended between neighboring pairs of NS5 branes. The fundamental matter is incorporated by including n semi-infinite D4 branes extending to the right and left in the x^6 direction.

It is easy to lift such a brane configuration to an M-theory curve [5]

$$F(t,v) \equiv p(v)t^3 + q(v)t^2 + r(v)t + s(v) = 0\,, \tag{3.1}$$

where $v = x^4 + ix^5$, $t = \exp\{(x^6 + ix^{10})/R\}$, x^{10} is the eleventh dimension of radius R. p, q, r and s are polynomials of degree n:

$$\begin{aligned} p &= \prod_{i=1}^{n}(v - m_i^{(1)} - M)\,, \\ q &= B_1 \cdot \prod_{j=1}^{n}(v - b_j - M)\,, \\ r &= B_2 \cdot \prod_{k=1}^{n}(v - a_k + M)\,, \\ s &= \prod_{\ell=1}^{n}(v - m_\ell^{(2)} + M)\,. \end{aligned} \tag{3.2}$$

The leading coefficients of p and s are set to 1 by rescaling t and v, and by a shift in v we set $\sum_k a_k = \sum_j b_j = 0$. Interpreting the positions in the v plane of the D4 branes as mass parameters or Coulomb branch vevs, we find that the $m_i^{(1)}$ and $m_\ell^{(2)}$ are the bare masses of the fundamentals of the first and the second SU factors, M is the bifundamental mass, and the traceless a_k and b_j are the eigenvalues of the adjoint vevs of the first and the second SU factors.

The B_i in (3.2) encode the gauge couplings through the relative asymptotic positions of the NS5 branes in the IIA picture. These positions are given by the roots of $F(t,v) = 0$ for large v, that is, the roots of $t^3 + B_1 t^2 + B_2 t + 1 = 0$. The relative positions of these roots are unaffected by the $\mathbb{Z}_3$ transformation of the coefficients B_i

$$(B_1, B_2) \to (\omega^p B_1, \omega^{2p} B_2)\,, \qquad p = 1, 2\,, \tag{3.3}$$

where $\omega = e^{2\pi i/3}$. Thus the space $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$ of inequivalent couplings that enters into the low-energy physics on the Coulomb branch of this $\mathrm{SU}(n) \times \mathrm{SU}(n)$ theory is the space $\mathbb{C}^2 \simeq \{B_1, B_2\}$ modded by the $\mathbb{Z}_3$ action (3.3). Furthermore, in addition to the $\mathbb{Z}_3$ orbifold fixed point at $B_1 = B_2 = 0$, this space has singularities whenever the asymptotic positions of the M5 branes collide — whenever $0 = 27 - 18B_1B_2 - B_1^2B_2^2 + 4B_1^3 + 4B_2^3$ — as well as weak-coupling singularities whenever one of the NS5 branes goes off to infinity: $B_1 \to \infty$ or $B_2 \to \infty$. Indeed, the space of $\mathrm{SU}\times\mathrm{SU}$ couplings can be parameterized by the $\mathbb{Z}_3$-invariant combinations $f_1 \equiv B_1/B_2^2$ and $f_2 \equiv B_2/B_1^2$, which have been chosen to correspond to the normalization of the SU(2) coupling f used in (2.1), so that they are related to gauge couplings at weak coupling as $\{f_1, f_2\} \sim \{e^{i\pi\tau_1}, e^{i\pi\tau_2}\}$.

We can check this identification of the coupling parameters (as well as our implicit identification of the vevs and bare masses in the $\mathrm{SU}\times\mathrm{SU}$ curve) by decoupling one of the SU factors by taking one of the NS5 branes off to infinity. For example, we can decouple the first SU factor by setting $B_2 = f_2 B_1^2$ with f_2 finite, and sending $B_1 \to \infty$. The SU $\times$ SU curve (3.1) then becomes, after rescaling $t \to B_1 t$ and dividing by B_1^3,

$$0 = t\left(p(v)t^2 + \frac{q(v)}{B_1}t + \frac{r(v)}{B_1^2}\right). \tag{3.4}$$

The overall factor of t is for the decoupled brane, and the remaining polynomial becomes, using (3.2),

$$0 = \prod_{i=1}^{n}\left(v - m_i^{(1)} - M\right)\cdot t^2 + \prod_{j=1}^{n}(v - b_j - M)\cdot t + f_2 \cdot \prod_{k=1}^{n}(v - a_k + M)\,. \tag{3.5}$$

Multiplying by $\prod_{i=1}^{n}(v - m_i^{(1)} - M)$, changing variables to $y = 2t\prod_{i=1}^{n}(v - m_i^{(1)} - M) + \prod_{j=1}^{n}(v - b_j - M)$, shifting $v \to v + M$, and identifying $M_i = m_i^{(1)}$ for $i = 1, \ldots, n$ and $M_i = a_{i-n} - 2M$ for $i = n+1, \ldots, 2n$, gives the scale invariant SU curve found in [3].

3.2 SU $\times$ Sp curves

Consider the scale invariant $\mathrm{SU}(2n) \times \mathrm{Sp}(2n)$ theory with one hypermultiplet in the bifundamental, $2n$ in the $\mathrm{SU}(2n)$ fundamental, and 2 in the $\mathrm{Sp}(2n)$ fundamental. This can be realized as a IIA brane configuration in the presence of an O6 orientifold plane of negative RR charge [8]. The O6$^-$ plane is the fixed point of a $\mathbb{Z}_2$ quotient which acts on the space-time coordinates as $x^{4,5,6} \to -x^{4,5,6}$, and thus extends along the $x^{0\cdots 3}$ and $x^{7\cdots 9}$ directions, and is located at $x^{4\cdots 6} = 0$. It is convenient to work on the double cover, by including mirror images for all branes, where the O6$^-$ plane has RR charge -8 in D6 brane units. The $\mathrm{SU}(2n) \times \mathrm{Sp}(2n)$ gauge theory is then constructed by placing two NS5 branes (and their mirror images) along the $x^{0\cdots 5}$ directions separated in x^6 but located at equal values in $x^{7\cdots 9}$, and $2n$ D4 branes along the $x^{0\cdots 3}$ and x^6 directions suspended between neighboring pairs of NS5 branes. The fundamental matter is incorporated by including D6 branes parallel to the O6$^-$ plane: two between the O6$^-$ plane and the first NS5 brane, and $2n$ between the two NS5 branes (as well as their mirror images).

Following [13], we can derive the curve for this brane configuration by first moving the D6 branes to left and right infinity, whereupon they drag D4 branes behind them upon passing through any NS5 branes [14]. Also, we can represent the O6$^-$ plane as a "neutral" O6 plane by pulling in 2 D6 branes (and their mirror images) from infinity to cancel the O6$^-$ plane RR charge. Upon passing through the NS5 branes, the D6 branes create two D4 branes between the NS5 branes and four between the NS5 brane and the O6$^-$ plane (as well as their mirror images). Thus the final configuration is simply four NS5 branes crossed by $2n + 4$ infinite D4 branes, all arranged symmetrically with respect to the origin of $x^{4\cdots 6}$.

It is easy to lift such a brane configuration to the M-theory curve

$$F(t,v) \equiv p(v)t^4 + q(v)t^3 + r(v)t^2 + q(-v)t + p(-v) = 0\,, \tag{3.6}$$

where $v = x^4 + ix^5$, $t = \exp\{(x^6 + ix^{10})/R\}$, x^{10} is the eleventh dimension of radius R; p, q and r are polynomials of degree $2n+4$, $r(v) = r(-v)$, and the $\mathbb{Z}_2$ identification is lifted to

$$(v,t) \to \left(-v, \frac{1}{t}\right). \tag{3.7}$$

The condition that there be an O6$^-$ plane implies [8] that this curve be non-singular on the Atiyah-Hitchin space. As discussed in [13], this in turns implies that $(\partial^\ell F/\partial v^\ell)|_{v=0}$ has a zero of order $4-\ell$ at $t=-1$ for $\ell = 0,\ldots,3$, giving

$$\begin{aligned}
p &= \prod_{i=1}^{2}(v-m_i)^2 \cdot \prod_{j=1}^{2n}(v-\mu_j - M),\\
q &= 4p[0] + 2vp[1] + A_1 \cdot v^2 \cdot \prod_{i=1}^{2}(v-m_i) \cdot \prod_{k=1}^{2n}(v - a_k - M),\\
r &= 6p[0] + 2v^2(q[2]-p[2]) + A_2 \cdot v^4 \cdot \prod_{\ell=1}^{n}(v^2 - b_\ell^2)\,,
\end{aligned} \tag{3.8}$$

where $p[n]$ refers to the coefficient of v^n in $p(v)$. Interpreting the positions in the v plane of the D4 branes as mass parameters or Coulomb branch vevs, we find that the m_i are the bare masses of the two Sp fundamentals, μ_j are the masses of the SU fundamentals, M is the bifundamental mass, the traceless a_k are the eigenvalues of the SU adjoint vev, and the b_ℓ likewise for the Sp adjoint vev.

The A_i in (3.8) encode the gauge couplings through the relative asymptotic positions of the NS5 branes in the IIA picture. These positions are given by the roots of $F(t,v) = 0$ for large v, that is, the roots of $t^4 + A_1t^3 + A_2t^2 + A_1t + 1 = 0$. The relative positions of these roots are unaffected by the $\mathbb{Z}_2$ transformation of the A_i coefficients

$$(A_1, A_2) \to (-A_1, A_2)\,. \tag{3.9}$$

Thus the space $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$ of inequivalent couplings that enters into the low-energy physics on the Coulomb branch of this $\mathrm{SU}(2n) \times \mathrm{Sp}(2n)$ theory is the space $\mathbb{C}^2 \simeq \{A_1, A_2\}$ modded by the $\mathbb{Z}_2$ action (3.9). Furthermore, in addition to the line of $\mathbb{Z}_2$ orbifold fixed points at $A_1 = 0$, this space has strong coupling singularities whenever the asymptotic positions of the M5 branes collide, which is when $A_2 + 2 = \pm 2A_1$ or $A_1^2 = 4A_2 - 8$, as well as weak coupling singularities whenever one of the M5 branes goes off to infinity: $A_1 \to \infty$ or $A_2 \to \infty$. Indeed, the space of $\mathrm{SU} \times \mathrm{Sp}$ couplings can be parameterized by the $\mathbb{Z}_2$-invariant combinations $g_1 \equiv A_2/A_1^2$ and $g_2 \equiv A_1^2/A_2^2$, which have been chosen to correspond to the normalization of the SU and Sp couplings used in the last section, so that they are related to gauge couplings at weak coupling as $\{g_1, g_2\} \sim \{e^{i\pi\tau_1}, e^{i\pi\tau_2}\}$ where τ_1 is the SU coupling and τ_2 the Sp coupling.

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We can check this identification of the coupling parameters (as well as our implicit identification of the vevs and bare masses) in the SU $\times$ Sp curve by decoupling the Sp factor (g_1 fixed, $A_i \to \infty$) or the SU factor (g_2 fixed, $A_i \to \infty$). Decoupling the Sp factor leads to considerations very similar to those discussed above in the case of the SU $\times$ SU curve, so we consider only the decoupling of the SU factor. This decoupling is also interesting since it involves passing from the $\{v, t\}$ space which is a double cover of the orbifold space, to the single-valued coordinates which resolve the orbifold singularity appropriately. We will need to do the same change of variables on the SU(2) $\times$ Sp(2) curve in the next subsection.

The SU $\times$ Sp curve (3.6) then becomes

$$0 = t\left(q(v)t^2 + r(v)t + q(-v)\right). \tag{3.10}$$

The overall factor of t is for the decoupled brane, and the remaining polynomial becomes

$$0 = \sqrt{g_2}\prod_{i=1}^{2n+2}(v - M_i)\cdot t + \left[2\sqrt{g_2}\prod_{i=1}^{2n+2} M_i + v^2\prod_{\ell=1}^{n}(v^2 - b_\ell^2)\right] + \sqrt{g_2}\prod_{i=1}^{2n+2}(v + M_i)\cdot\frac{1}{t}, \tag{3.11}$$

where we have used (3.8), divided by $A_2 v^2/t$, and defined $M_i = m_i$ for $i = 1, 2$ and $M_i = a_{i-2}$ for $i = 3, \ldots, 2n+2$. In order to compare this curve with previously derived genus-n Sp($2n$) curves, we must divide out the orbifold identifications (3.7). To do this, define the invariant coordinates

$$\begin{aligned} x &= v^2 \\ y &= \left[t - \frac{1}{t}\right] v^{-1} \\ z &= \left[t + \frac{1}{t} + 2\right] v^{-2}, \end{aligned} \tag{3.12}$$

which are related by

$$y^2 = xz^2 - 4z\,. \tag{3.13}$$

Note that the change of variables (3.12) is singular when $v = 0$; it serves to resolve the orbifold singularities at $v = 0$, $t = \pm 1$ so that the resulting space has the complex structure of the Atiyah-Hitchin space [15], which is the appropriate M-theory resolution of the O6$^-$ plane singularity [16]. In these variables, the curve (3.11) becomes

$$0 = xP_0(x)\cdot z + xP_1(x)\cdot y - 2P_0(x) + 2\sqrt{g_2}\prod_{i=1}^{2n+2} M_i + x\prod_{\ell=1}^{n}(x - b_\ell^2), \tag{3.14}$$

where we have defined P_0 and P_1 by $\sqrt{g_2}\prod_{i=1}^{2n+2}(v - M_i) = P_0(v^2) + vP_1(v^2)$. Making the change of variables $\widetilde{y} = P_0 y + xP_1 z - 2P_1$ and $\widetilde{z} = xP_1 y + xP_0 z - 2P_0$, (3.13) and (3.14) become

$$\widetilde{z} = -2\sqrt{g_2}\prod_{i=1}^{2n+2} M_i - x\prod_{\ell=1}^{n}(x - b_\ell^2)\,,$$

$$x\widetilde{y}^2 = \widetilde{z}^2 - 4g_2\prod_{i=1}^{2n+2}(x - M_i^2)\,, \tag{3.15}$$

where we have used the identity $P_0^2 - xP_1^2 = g_2\prod_{i=1}^{2n+2}(x - M_i^2)$. Eliminating $\widetilde{z}$ in (3.15) then gives the $\mathrm{Sp}(2n)$ curve found in [4].

3.3 $\mathrm{SU}(2)\times\mathrm{SU}(2)$ and $\mathrm{SU}(2)\times\mathrm{Sp}(2)$ scale invariant curves

We now specialize to the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ and $\mathrm{SU}(2)\times\mathrm{Sp}(2)$ theories which are of interest for the S-duality argument.

Consider first the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ scale invariant theory with zero bare masses for the hypermultiplets. From (3.1) and (3.2) the Coulomb branch of this theory is described by

$$t^3v^2 + B_1t^2(v^2 - u_1) + B_2t(v^2 - u_2) + v^2 = 0\,, \tag{3.16}$$

where $u_1 = -b_1b_2$ and $u_2 = -a_1a_2$ denote the Coulomb branch moduli of the two $\mathrm{SU}(2)$'s. To study degenerations of (3.16) on the Coulomb branch it is convenient to represent it as a double cover of the complex t plane:

$$v^2 = \frac{t(B_1u_1t + B_2u_2)}{(t^3 + B_1t^2 + B_2t + 1)}\,. \tag{3.17}$$

The change of variables

$$y = (t^3 + B_1t^2 + B_2t + 1)v \tag{3.18}$$

takes (3.17) to the hyperelliptic form

$$y^2 = t(B_1u_1t + B_2u_2)(t^3 + B_1t^2 + B_2t + 1)\,. \tag{3.19}$$

We pause here to discuss the validity of changes of variables like (3.18), which we will use again below on the $\mathrm{SU}(2)\times\mathrm{Sp}(2)$ curve, and which we also used in the decoupling checks of the last subsections. It is important that the complex structures of curves related by these changes of variables are the same since we will match the parameters of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ and $\mathrm{SU}(2)\times\mathrm{Sp}(2)$ curves by comparing the complex structures of their hyperelliptic forms. The issue is the apparent singularity of the change of variables (3.18) whenever $t^3 + B_1t^2 + B_2t + 1 = 0$. In fact this change of variables, when properly understood, is not singular on the curve, and so the resulting hyperelliptic curve (3.19) is equivalent to (has the same complex structure as) the prior curve (3.17).

The key point lies in the treatment of the points at infinity on the curves. Let us generalize to a situation where we have a curve of the form

$$v^2 \prod_{j=1}^{m} (t - f_j) = \prod_{i=1}^{m} (t - e_i)\,, \tag{3.20}$$

which we would like to think of as representing a Riemann surface of genus $m-1$. Thought of as a curve embedded in $\mathbb{C}^2 = \{v,t\}$, though, (3.20) is non-compact, going off to infinity as $t \to f_j$ and $t \to \infty$. We can compactify this curve by replacing the $\{v,t\}$ space with an appropriate projective space; the correct choice of projective space is determined by demanding that the genus of the resulting compact surface indeed be $m-1$. This is achieved if each infinity $t \to f_i$ is replaced by a single point, while the $t \to \infty$ infinity is compactified at two distinct points. The appropriate projective space which does this is the direct product of two Riemann spheres, $\mathbb{P}^1 \times \mathbb{P}^1$, which can be defined as $\mathbb{C}^4 = \{u,v,s,t\}$ modulo the identifications $\{u,v,s,t\} \simeq \{u,v,\lambda s,\lambda t\}$ for $\lambda \in \mathbb{C}^*$, and $\{u,v,x,z\} \simeq \{\mu u, \mu v, s, t\}$ for $\mu \in \mathbb{C}^*$. The curve is homogenized to $v^2 \prod_{j=1}^m (t - f_j s) = u^2 \prod_{i=1}^m (t - e_i s)$. The infinities of the $\{v,t\} = \mathbb{C}^2$ space are compactified to two (intersecting) copies of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$, while the homogeneous curve intersects these "infinities" at the points $\{u,v,s,t\} = \{0,1,1,f_j\}$ (corresponding to $t \to f_j$) and $\{1,\pm 1,0,1\}$ (corresponding to $t \to \infty$).

We are interested in the change of variables $y = v \cdot \prod_{j=1}^m (t - f_j)$ which in homogeneous coordinates can be written $y = (v/u) \cdot \prod_{j=1}^m (t - f_j s)$, $x = t$, $z = s$. The $\mathbb{P}^1 \times \mathbb{P}^1$ identifications on $\{u,v,s,t\}$ imply $\{y,x,z\} \simeq \{\lambda^m y, \lambda x, \lambda z\}$ for $\lambda \in \mathbb{C}^*$, which defines a point in the weighted projective space $\mathbb{P}^2_{(m,1,1)}$. This is a smooth space except for a $\mathbb{Z}_m$ orbifold singularity at the point $\{y,x,z\} = \{1,0,0\}$. The change of variables thought of as a map from $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2_{(m,1,1)}$ is singular on the $\mathbb{P}^1$ at $v = \infty$ which is mapped to the $\mathbb{Z}_m$ orbifold point of $\mathbb{P}^2_{(m,1,1)}$, except for the points $\{v,t\} = \{\infty, f_i\}$ which are blown up to the $\mathbb{P}^1$ of points $\{y,x,z\} = \{\cdot, f_i, 1\}$.

The image of the homogeneous curve under this change of variables is the genus $m-1$ hyperelliptic curve

$$y^2 = \prod_{i=1}^{m} (x - e_i z)(x - f_i z)\,, \tag{3.21}$$

which does not intersect the $\mathbb{Z}_m$ orbifold point of $\mathbb{P}^2_{(m,1,1)}$ if $\prod_i e_i f_i \neq 0$. In particular, the $\mathbb{P}^1 \times \mathbb{P}^1$ curve approaches the points $\{u,v,s,t\} = \{0,1,1,f_j\}$ in such a way that their images in $\mathbb{P}^2_{(m,1,1)}$ miss the orbifold point. Therefore the change of variables is a holomorphic mapping between the abstract Riemann surfaces, and so equates their complex structures.

In the case of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ curve (3.17) the f_i are roots of $t^3 + B_1 t^2 + B_2 t + 1 = 0$, while the e_i are 0, $-(B_2 u_2)/(B_1 u_1)$, and ∞. The branch points at zero and infinity are harmless as can be seen by the fact that an $\mathrm{SL}(2,\mathbb{C})$ transformation on the $\{s,t\}$ $\mathbb{P}^1$ preserves the complex structure of the curve and can be used to move all branch points to finite points on the t plane.

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We return now to discuss the $\mathrm{SU}(2) \times \mathrm{Sp}(2)$ theory. From (3.6) and (3.8) the curve of the scale invariant $\mathrm{SU}(2) \times \mathrm{Sp}(2)$ theory with zero hypermultiplet masses is given by

$$v^2 t^4 + A_1(v^2 - v_1)t^3 + A_2(v^2 - v_2)t^2 + A_1(v^2 - v_1)t + v^2 = 0\,, \tag{3.22}$$

where $v_1 = -a_1 a_2$ and $v_2 = b_1^2$ are Coulomb branch moduli of the SU and Sp factor respectively. This curve is of the form (3.20) with $m = 4$ (and one f_j at infinity), thus describing a genus 3 Riemann surface (as is also clear from its brane construction). It was supposed to be equivalent to the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ curve, which was genus 2. The reason for the mismatch is that the $\mathrm{SU}(2) \times \mathrm{Sp}(2)$ curve was constructed on the double cover of the $\mathrm{O6}^-$ plane orbifold space.

Changing to single-valued variables on the orbifold space *via* (3.12), which parameterize the non-singular Atiyah-Hitchin space [15] (the M theory resolution of the space transverse to the $\mathrm{O6}^-$ plane [16]), gives the curve (3.22) as the intersection of the surfaces

$$\begin{aligned} y^2 &= xz^2 - 4z\,, \\ 0 &= x((xz-2)^2 - 2) + A_1(x - v_1)(xz - 2) + A_2(x - v_2)\,. \end{aligned} \tag{3.23}$$

Change variables by $s = xz - 2$, leaving x and y unchanged. Then the curve becomes the intersection

$$\begin{aligned} xy^2 &= s^2 - 4\,, \\ 0 &= x(s^2 - 2) + A_1(x - v_1)s + A_2(x - v_2)\,. \end{aligned} \tag{3.24}$$

This change of variables is singular at $x = 0$ which is a direction at infinity on the curve. As in the discussion above, as long as we treat the "points" at infinity correctly so as to preserve the genus of the curve, the complex structure will be preserved by the change of variables. Eliminating x from (3.24) gives the curve

$$y^2 = \frac{(s^2 - 4)(s^2 + A_1 s + A_2 - 2)}{(A_1 v_1 s + A_2 v_2)}\,. \tag{3.25}$$

(x was the right variable to eliminate since only x is single valued on the Atiyah-Hitchin space, which is a double cover of the y-z plane.) Finally, by the type of change of variables discussed above, $w = (A_1 v_1 s + A_2 v_2)y$, the genus 2 curve emerges in the hyperelliptic form

$$w^2 = (A_1 v_1 s + A_2 v_2)(s^2 - 4)(s^2 + A_1 s + A_2 - 2)\,. \tag{3.26}$$

Since the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and $\mathrm{SU}(2) \times \mathrm{Sp}(2)$ theories are physically identical, the two genus 2 hyperelliptic curves (3.19) and (3.26) must have the same complex structure as a function of the couplings and vevs. Thus there must be an $\mathrm{SL}(2, \mathbb{C})$

transformation relating t and s which maps the branch points of (3.19) to those of (3.26). If we map the branch points at infinity to each other, and the branch point at $s = -2$ to the one at $t = 0$, then we must find a linear transformation $4\beta t = s + 2$ and a map between the vevs and couplings which satisfies

$$(A_1 v_1 s + A_2 v_2)(s-2)(s^2 + A_1 s + A_2 - 2) \propto (B_1 u_1 t + B_2 u_2)(t^3 + B_1 t^2 + B_2 t + 1)\,, \tag{3.27}$$

for some β. Since the theory is scale-invariant, we can choose an arbitrary relative scaling of the u and v vevs so that $u_1 = v_1$. We then find the following relations between couplings,

$$\begin{aligned} A_1 &= 8 + 4\beta B_1\,, \\ A_2 &= 30 + 24\beta B_1 + 16\beta^2 B_2\,, \\ 0 &= 1 + \beta B_1 + \beta^2 B_2 + \beta^3\,, \end{aligned} \tag{3.28}$$

while the vevs are related by $v_1 = u_1$ and $(A_2/A_1)v_2 = 2u_1 + 4\beta(B_2/B_1)u_2$. These matching relations are the main result of this section. They can be inverted to read

$$\begin{aligned} B_1 &= \frac{A_1 - 8}{4\alpha}\,, \\ B_2 &= \frac{A_2 - 6A_1 + 18}{16\alpha^2}\,, \\ 16\alpha^3 &= 2A_1 - A_2 - 2\,, \end{aligned} \tag{3.29}$$

for the couplings, with the vevs related by $u_1 = v_1$ and $(B_2/B_1)u_2 = [(A_2/A_1)v_2 - 2v_1]/(4\alpha)$, corresponding to a map $4\alpha t = s + 2$ between the curves.

Finally, one can easily check that in the weak coupling limits, the above matching of parameters reduces to the appropriate identifications. For example, decoupling the SU(2) factor of the SU(2) $\times$ Sp(2) theory by sending $A_i \to \infty$ keeping $g_2 = A_1^2/A_2^2$ fixed (and thus $g_1 \to 0$), (3.29) implies that the SU(2) $\times$ SU(2) couplings go as

$$\begin{aligned} f_1 &\equiv \frac{B_1}{B_2^2} \to \frac{4\sqrt{g_2}(1 + 2\sqrt{g_2})}{(1 + 6\sqrt{g_2})^2}\,, \\ f_2 &\equiv \frac{B_2}{B_1^2} \to 0\,, \end{aligned} \tag{3.30}$$

which recovers precisely the mapping (2.4) between the SU(2) and Sp(2) couplings used in section 2, and is a non-trivial consistency check on the calculations of this section.

3.4 Summary of SU(2) $\times$ SU(2) low energy coupling spaces

We now summarize what we have just derived about the space of couplings of the SU(2) $\times$ SU(2) theory as they appear in the low energy effective actions on the Coulomb branch described by the SU(2) $\times$ SU(2) and SU(2) $\times$ Sp(2) curves. We denote these two spaces of couplings by $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$ and $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$ respectively.

3.4.1 $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$

The SU(2) × SU(2) low energy effective action is described by two complex couplings B_1 and B_2 which parameterize an $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}} \simeq \mathbb{C}^2/S_3$ orbifold space. The S_3 orbifold identifications are generated by the $\mathbb{Z}_3$ element

$$\mathcal{P} : (B_1, B_2) \to (\omega B_1, \omega^2 B_2)\,, \tag{3.31}$$

where ω is a cube root of unity, as well as by the $\mathbb{Z}_2$ element

$$\mathcal{Q} : (B_1, B_2) \to (B_2, B_1) \tag{3.32}$$

which simply interchanges the two SU(2) factors. Resulting from the S_3 identifications, $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$ has three lines of $\mathbb{Z}_2$ orbifold singularities when $B_1 = \omega B_2$ which intersect in an S_3 orbifold point at $B_1 = B_2 = 0$. $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$ also has strong-coupling singularities when

$$0 = 27 - 18B_1B_2 - B_1^2B_2^2 + 4B_1^3 + 4B_2^3 \tag{3.33}$$

as well as weak-coupling singularities when $B_1 \to \infty$ or $B_2 \to \infty$. The $\mathbb{Z}_3$-invariant couplings

$$f_1 \equiv \frac{B_1}{B_2^2} \qquad \text{and} \qquad f_2 \equiv \frac{B_2}{B_1^2}\,, \tag{3.34}$$

are related to the $\{\tau_1, \tau_2\}$ gauge couplings of the two SU(2) factors by $\{f_1, f_2\} \sim \{e^{i\pi\tau_1}, e^{i\pi\tau_2}\}$ at weak coupling.

3.4.2 $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$

The SU(2) × Sp(2) curve, though describing the same theory, has a very different space of couplings, A_1 and A_2, parameterizing the orbifold space $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}} \simeq \mathbb{C}^2/\mathbb{Z}_2$. The $\mathbb{Z}_2$ identification acts as

$$\mathcal{R} : (A_1, A_2) \to (-A_1, A_2)\,, \tag{3.35}$$

and gives rise to a line of $\mathbb{Z}_2$ orbifold fixed points in $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$ when $A_1 = 0$. In addition, $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$ has strong coupling singularities when

$$A_2 + 2 = \pm 2A_1 \qquad \text{or} \qquad A_1^2 = 4A_2 - 8\,, \tag{3.36}$$

as well as weak-coupling singularities when $A_1 \to \infty$ or $A_2 \to \infty$. The $\mathbb{Z}_2$-invariant couplings

$$g_1 \equiv \frac{A_2}{A_1^2} \qquad \text{and} \qquad g_2 \equiv \frac{A_1^2}{A_2^2}\,, \tag{3.37}$$

are related to the $\{\tau_1, \tau_2\}$ gauge couplings of the SU(2) and Sp(2) factors, respectively, by $\{g_1, g_2\} \sim \{e^{i\pi\tau_1}, e^{i\pi\tau_2}\}$ at weak coupling.

3.4.3 $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}} \leftrightarrow \mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$ map

Finally, the low energy $\mathrm{SU}(2)\times\mathrm{SU}(2)$ and $\mathrm{SU}(2)\times\mathrm{Sp}(2)$ descriptions of the theory are found to be equivalent as long as the parameters of one theory are mapped to those of the other by $\mathcal{T}:\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}\to\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$ defined by

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \mathcal{T}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \equiv \begin{pmatrix} (A_1-8)/(4\alpha) \\ (A_2-6A_1+18)/(16\alpha^2) \end{pmatrix}, \tag{3.38}$$

where

$$16\alpha^3 = 2A_1 - A_2 - 2\,, \tag{3.39}$$

or its inverse

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \mathcal{T}^{-1}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \equiv \begin{pmatrix} 8+4\beta B_1 \\ 30+24\beta B_1+16\beta^2 B_2 \end{pmatrix}, \tag{3.40}$$

with

$$0 = 1+\beta B_1+\beta^2 B_2+\beta^3\,. \tag{3.41}$$

4. S-duality in the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ theory

We will now derive the enlarged S-duality group of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ theory. The idea is a straightforward generalization of the strategy used for a single $\mathrm{SU}(2)$ factor reviewed in section 2: the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ model can be reached by flowing down from both the $\mathrm{SU}(n)\times\mathrm{SU}(n)$ and $\mathrm{SU}(2n)\times\mathrm{Sp}(2n)$ series. Denoting by $\mathcal{M}$ the true coupling space of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ theory, we therefore expect to find some multiple cover $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$ of $\mathcal{M}$ as the coupling space realized by flowing down in the $\mathrm{SU}(n)\times\mathrm{SU}(n)$ series, and a different multiple cover $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$ by flowing down in the $\mathrm{SU}(2n)\times\mathrm{Sp}(2n)$ series. We then use the equivalence of the two descriptions of the theory to deduce a map identifying $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$ with $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$. If this map is not a simple one-to-one map, then we thereby deduce extra identifications leading to the "smaller" coupling space $\mathcal{M}$ and therefore a larger S-duality group $\pi_1(\mathcal{M})$.

The determination of $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$ and $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$ is easy, as we have already done it in [10]. There we showed that the embedding argument leads to a coupling space for the $\mathrm{SU}(n)\times\mathrm{SU}(n)$ theory which is precisely the $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$ described above in eqns. (3.31)–(3.33), and likewise that the $\mathrm{SU}(2n)\times\mathrm{Sp}(2n)$ theory coupling space is the $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$ described above in eqns. (3.35)–(3.36). The map between $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$ and $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$ is then the one derived at length in the last section, and summarized in eqns. (3.38)–(3.41). As this map is obviously not one-to-one, we have therefore found new S-duality identifications on the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ coupling space, which is what we aimed to show.

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The remainder of this section will be devoted to understanding some properties of the $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}} \leftrightarrow \mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}}$ map $\mathcal{T}$ (3.38)–(3.41), and thereby of the the resulting enlarged S-duality group, $\Gamma = \pi_1(\mathcal{M})$. We will refer to the $\mathbb{C}^2$ of B_i parameters as $\mathbb{C}^2_B$ and of A_i parameters $\mathbb{C}^2_A$. To see algebraically the extra identifications induced on $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}}$ by the map $\mathcal{T} : \mathbb{C}^2_A \to \mathbb{C}^2_B$, we use it to construct maps from $\mathbb{C}^2_B$ to itself. Note first that $\mathcal{T}$ and $\mathcal{T}^{-1}$ each have three image points corresponding to the three different values that α or β can take. In the case of $\mathcal{T}$, the three α's differ only by cube root of unity phases, and the three image points in $\mathbb{C}^2_B$ are related by the $\mathbb{Z}_3$ identification $\mathcal{P}$ (3.31). One the other hand, the image of $\mathcal{T}^{-1}$ is generically three distinct points in $\mathbb{C}^2_A$ unrelated by the $\mathbb{Z}_2$ identification $\mathcal{R}$ (3.35). However, the images under $\mathcal{T}^{-1}$ of three points in $\mathbb{C}^2_B$ related by $\mathcal{P}$ are all the same three points in $\mathbb{C}^2_A$, since a $\mathcal{P}$ action on the B_i just rotates the roots of (3.41), leaving (3.40) invariant.

Since the $\mathcal{T}$ map commutes with $\mathcal{P}$, we can formulate the identifications directly on $\mathbb{C}^2_f \equiv \mathbb{C}^2_B/\{\mathcal{P}\}$ with coordinates f_i given by (3.34). In these variables $\mathcal{T}$ becomes

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathcal{T}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 4(A_1-8)(2A_1-A_2-2)(A_2-6A_1+18)^{-2} \\ (A_2-6A_1+18)(A_1-8)^{-2} \end{pmatrix}, \tag{4.1}$$

and $\mathcal{T}^{-1}$ reads

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \mathcal{T}^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} (8f_2+4\gamma)/f_2 \\ (30f_2+24\gamma+16\gamma^2)/f_2 \end{pmatrix}, \tag{4.2}$$

where γ is a root of

$$0 = f_1\gamma^3 + \gamma^2 + \gamma + f_2 \,. \tag{4.3}$$

Note that while $\mathcal{T}$ is a single map, $\mathcal{T}^{-1}$ is generically three maps, one for each γ satisfying (4.3). Nevertheless, it is easy to check that $\mathcal{T}\cdot\mathcal{T}^{-1}$ maps points in $\mathbb{C}^2_f$ to themselves, and it follows that repeated applications of $\mathcal{T}$ and $\mathcal{T}^{-1}$ generate no further identifications between $\mathbb{C}^2_A$ and $\mathbb{C}^2_f$.

Since $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{SU}} \simeq \mathbb{C}^2_f/\{\mathcal{Q}\}$ and $\mathcal{M}_{\mathrm{SU}\cdot\mathrm{Sp}} \simeq \mathbb{C}^2_A/\{\mathcal{R}\}$, where $\mathcal{Q}$ and $\mathcal{R}$ are the $\mathbb{Z}_2$ identifications $\mathcal{Q} : f_1 \leftrightarrow f_2$, and $\mathcal{R} : A_1 \leftrightarrow -A_1$, further identifications will arise upon combining $\mathcal{T}$ with $\mathcal{R}$ and $\mathcal{Q}$. It is algebraically easiest to work on $\mathbb{C}^2_f$ where there are three generators of non-trivial maps involving $\mathcal{T}$, namely $\mathcal{S}_i \equiv \mathcal{T}\cdot\mathcal{R}\cdot\mathcal{T}^{-1}$. Explicitly, this map reads

$$\mathcal{S}_i\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} (4f_2+\gamma)(3f_2+2\gamma+\gamma^2)(6f_2+3\gamma+\gamma^2)^{-2} \\ f_2(6f_2+3\gamma+\gamma^2)(4f_2+\gamma)^{-2} \end{pmatrix}, \tag{4.4}$$

where γ is a root of (4.3). The subscript on $\mathcal{S}$ denotes the three different choices of roots for γ, which lead generically to three different image points in $\mathbb{C}^2_f$. Thus the new S-duality identifications $\mathcal{S}_i$ that we have found imply at least a four-fold identification on $\mathbb{C}^2_f$ (the original point and its three images). Furthermore, the orbit of a given point under repeated applications of the $\mathcal{S}_i$ can be shown to be just this set of four points. The $\mathbb{Z}_2$ identification $\mathcal{Q}$ on $\mathbb{C}_f^2$ does not commute with the $\mathcal{S}_i$, though it can be shown that for a given $\mathcal{S}_i$, there exist an $\mathcal{S}_j$ and an $\mathcal{S}_k$ such that $\mathcal{S}_i\mathcal{Q}\mathcal{S}_j\mathcal{Q}\mathcal{S}_k\mathcal{Q} = 1$. The minimum orbit of a generic point satisfying these relations comprises 20 points, as shown in figure 1. In fact this is the generic orbit in $\mathbb{C}_f^2$ under the complete set of identifications generated by $\mathcal{S}_i$ and $\mathcal{Q}$, as checked numerically.

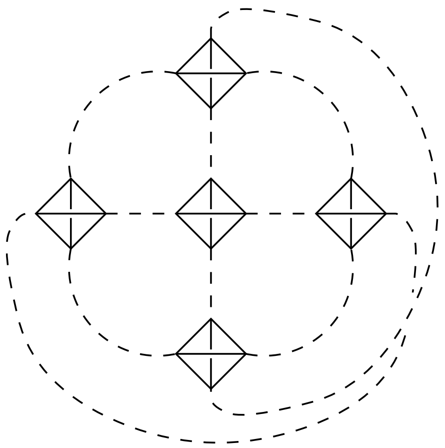

Figure 1: The generic orbit of a point in $\mathbb{C}_f^2$ under $\{\mathcal{S}_j, \mathcal{Q}\}$. The solid lines (edges of the tetrahedra) denote the action of the $\mathcal{S}_j$ maps, while the dashed lines connecting the tetrahedra denote the action of the $\mathcal{Q}$ map.

In summary, because of the algebraic complexity of the $\mathcal{S}_i$ generators, we have been unable to find a simpler description of the resulting true coupling space $\mathcal{M}$ than

$$\mathcal{M} \simeq \mathbb{C}_f^2/\{\mathcal{Q}, \mathcal{S}_i\} \tag{4.5}$$

with punctures at points satisfying (3.33) which reads in the f_i coordinates:

$$0 = 1 - 4f_1 - 4f_2 + 18f_1f_2 - 27f_1^2f_2^2\,, \tag{4.6}$$

as well as weak coupling singularities when $f_1f_2 = 0$. For clarity, we emphasize that $\pi_1(\mathcal{M})$ — the S-duality group of $\mathcal{M}$ — is *not* just the group generated by $\mathcal{Q}$ and $\mathcal{S}_i$. There are many reasons for this: $\mathbb{C}_f^2$ already has $\mathbb{Z}_3$ orbifold points at $f_1 = f_2 = 0$ and $f_1 = f_2 = \infty$; $\mathcal{Q}$ and $\mathcal{S}_i$ act with fixed points; there are also strong and weak coupling punctures on $\mathbb{C}_f^2$; finally, $\mathcal{Q}$ and $\mathcal{S}_i$ do not even generate a group since there is no consistent labeling of the $\mathcal{S}_i$ — the three roots of (4.3) — on the whole of $\mathbb{C}_f^2$.

We can, however, argue that $\mathcal{M}$ is *not* just the cartesian product of two copies of the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ as one might naively have guessed. If it were this product, $\mathcal{M}$ would have (complex) lines of $\mathbb{Z}_3$ fixed points, whereas it is straightforward to check that $\mathcal{Q}$ and $\mathcal{S}$ only have isolated $\mathbb{Z}_3$ fixed points which occur when (f_1, f_2) is one of

$$\left(\frac{1}{3},0\right),\ \left(0,\frac{1}{3}\right),\ \left(\frac{1}{3},\frac{1}{3}\right),\ \left(\frac{37+i\sqrt{3}}{98},\frac{37-i\sqrt{3}}{98}\right),\ \left(\frac{37-i\sqrt{3}}{98},\frac{37+i\sqrt{3}}{98}\right). \tag{4.7}$$

In fact, these five points are all identified under $\mathcal{Q}$ and $\mathcal{S}_i$, so there is only a single $\mathbb{Z}_3$ fixed point in $\mathcal{M}$.

Note that the first two entries in (4.7) are the $\mathbb{Z}_3$ points, identified in section 2, on the coupling space of one SU(2) factor in the limit where the other is decoupled.

The orbit of the $\mathbb{Z}_2$ points $f = 2/9, \infty$, of a single SU(2) factor also includes points at strong coupling:

$$\left(\frac{2}{9},0\right), \left(0,\frac{2}{9}\right), (\infty,0), (0,\infty), \left(\frac{1}{3},\frac{1}{3}\right), (-1,-1), \left(\frac{5}{16},\frac{8}{25}\right), \left(\frac{8}{25},\frac{5}{16}\right). \quad (4.8)$$

In fact, there are whole (complex) lines of $\mathbb{Z}_2$ fixed points. Though they are hard to characterize explicitly, they all seem to be images of the $\mathcal{Q}$ fixed line $f_1 = f_2$ under $\mathcal{S}_i$. These images intersect in an S_3 orbifold point whose orbit in $\mathbb{C}^2_f$ is

$$(\infty,\infty), \left(\frac{3}{8},\frac{1}{3}\right), \left(\frac{1}{3},\frac{3}{8}\right), \left(\frac{1}{2},\frac{1}{2}\right). \quad (4.9)$$

These examples illustrate the interesting feature of the $\mathcal{Q}$ and $\mathcal{S}_i$ maps that they equate "strong coupling" punctures — f_i satisfying (4.6) — with weak coupling punctures satisfying $f_1 f_2 = 0$. This is true in general: all strong coupling punctures are so identified with weak coupling points. To see this, note that f_1 and f_2 satisfy (4.6) precisely when two roots of (4.3) coincide. A double root of γ satisfies (4.3) and its first derivative: $0 = 3f_1\gamma^2 + 2\gamma + 1$. Rewriting this as $f_1\gamma^3 = -(2\gamma^2+\gamma)/3$ and substituting into (4.3) gives $\gamma^2 + 2\gamma + 3f_2 = 0$. But, by (4.4), this implies that $\mathcal{S}_i(f_1) = 0$ for this choice of the root γ, and thus that the strong coupling puncture is mapped to a weak coupling singularity. Thus S-duality identifications remove all "ultra-strong" coupling points from $\mathcal{M}$, just as in the case of the $SL(2,\mathbb{Z})$ duality of a single SU(2) factor.

Also, the point $(1/3, 1/3)$ is special as its image under $\mathcal{S}_i$ depends on how one approaches it. Generically, its image is the point $(1/3, 0)$, but if one approaches it along the particular direction $(f_1, f_2) = (1/3)\cdot(1+\epsilon, 1+\epsilon+k\epsilon^{3/2})$, then its image under S_i is the whole $(f, 0)$ plane, where f depends on k.

Finally, one should bear in mind that all our arguments only show that the extra identifications on $\mathbb{C}^2_f$ leading to $\mathcal{M}$ are necessary, but do not imply that there are no further identifications on $\mathcal{M}$. In principle one could rule out the existence of further identifications from the low energy effective action on the Coulomb branch by showing that the low energy data is different at distinct points of $\mathcal{M}$. Though this is beyond the scope of the present paper, one piece of evidence for there being no further identifications on $\mathcal{M}$ is the fact, checked above (3.30), that in the limit where one of the SU(2) factors decouples, $\mathcal{M}$ already encodes the full $SL(2,\mathbb{Z})$ S-duality group of the other SU(2) factor.

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